Black Box Scattering

7/14/2020

Previously:

(Convention: $\Delta := -\sum_j \partial_{x_j}^2$)

Potential scattering:

$$\boxed{\Delta + V - \lambda^2 = 0} \quad \text{on } \mathbb{R}^n$$

Motivated by wave equation

$$\boxed{(\partial_t^2 + \Delta + V)u = f}$$

Black Box Scattering

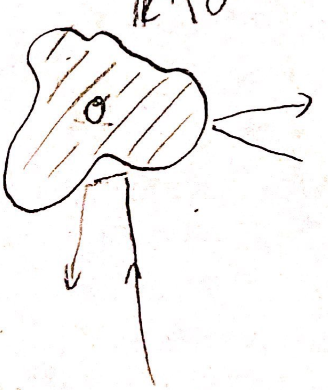
"What immediately suggests itself, then, is that these characteristic properties themselves be treated as the main object of investigation, by defining and dealing with abstract objects which need satisfy no other conditions than those required by the very theory to be developed."
– Constantin Carathéodory

Idea:

Generalize potential scattering to other situations where "scattering" happens

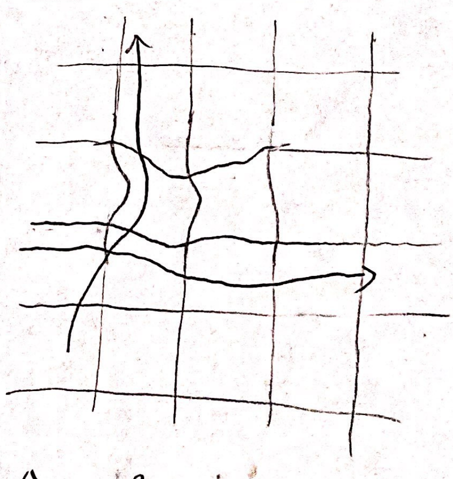
Examples:

Obstacle Scattering



$\Delta - \lambda^2 = 0$ on $\mathbb{R}^n \setminus \mathcal{O}$
(with boundary conditions)

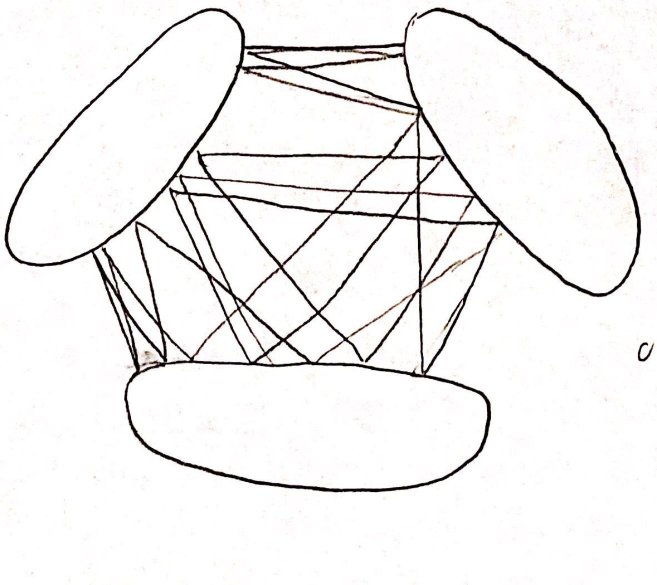
Metric Perturbation



$\Delta_g - \lambda^2 = 0$
$g_{ij} = \delta_{ij}$ outside compact set

Overview of Results

Potential Scattering	Black Box Scattering
Meromorphic Continuation of R_V (Yonah's/Haoren's talks)	Meromorphic Continuation of R
Resonance Upper Bound $N(r) \lesssim r^n$ (Izak's talk)	Resonance Upper Bound $N(r) \lesssim r^{n^\#}$
Rellich's Thm: No resonances in $\mathbb{R}\setminus\{0\}$ (James's talk)	Rellich's Thm: No resonances in $\mathbb{R}\setminus\{0\}$ except eigenvalues
Trace Formula for Resonances (Mitchell's 2nd talk)	Trace formula provided $\mathbb{1}_{B_{R_0}(0)}(P-i)^{-k}$ trace class for some k (Sjöstrand, Zworski, 1994)
Pole-Free Regions: Exist $A, C, \delta > 0$ s.t. $\{\operatorname{Im}\lambda \geq -A - \delta\log(1+\lvert\lambda\rvert)\} \cap \{\lvert\lambda\rvert > C\}$ pole-free (Mitchell's 1st talk)	Need "non-trapping" assumptions to establish pole-free regions
Resonance Expansion of Waves (Mitchell's 1st talk)	Tricky without pole-free regions. Subject of current research!



"trapping" in obstacle scattering

Basic Definitions

Let $\mathcal{H} := \underbrace{\mathcal{H}_{R_0}}_{\text{"black box"}} \oplus L^2(\mathbb{R}^n \setminus B_{R_0}(0))$ be Hilbert space

Orthogonal projections:

$$\mathbb{1}_{B_{R_0}(0)} : u \longmapsto u|_{B_{R_0}(0)} \in \mathcal{H}_{R_0}$$

$$\mathbb{1}_{\mathbb{R}^n \setminus B_{R_0}(0)} : u \longmapsto u|_{\mathbb{R}^n \setminus B_{R_0}(0)} \in L^2(\mathbb{R}^n \setminus B_{R_0}(0))$$

Note: $\mathcal{H}$ need not be $L^2(\mathbb{R}^n)$

Let $\mathcal{H}_0 := \{u \in \mathcal{H} : u|_{\mathbb{R}^n \setminus B_{R_0}(0)} \in L^2_0(\mathbb{R}^n \setminus B_{R_0}(0))$

$\mathcal{H}_{loc} := \mathcal{H}_{R_0} \oplus L^2_{loc}(\mathbb{R}^n \setminus B_{R_0}(0))$

Def. An unbounded self-adjoint operator $P : \mathcal{H} \to \mathcal{H}$ with domain $\mathcal{D} \subseteq \mathcal{H}$ is a black box Hamiltonian (BBH) if:

(a) $\mathbb{1}_{\mathbb{R}^n \setminus B_{R_0}(0)} \mathcal{D} \subseteq H^2(\mathbb{R}^n \setminus B_{R_0}(0))$

(b) $\mathbb{1}_{\mathbb{R}^n \setminus B_{R_0}(0)}(Pu) = \Delta(u|_{\mathbb{R}^n \setminus B_{R_0}(0)}) \quad \forall u \in \mathcal{D}$

(c) If $v \in H^2(\mathbb{R}^n)$, $v|_{B_{R_0+\varepsilon}(0)} \equiv 0$ for some $\varepsilon > 0$, then $v \in \mathcal{D}$

(d) $\mathbb{1}_{B_{R_0}(0)}(P+i)^{-1}$ is a compact operator

More Basic Definitions/Results

***Prop:** If $v \in H^2(\mathbb{R}^n)$, $v|_{B_{R_0+\varepsilon}(0)} \equiv 0$ for some $\varepsilon > 0$, then
$Pv = \Delta v$

Proof:
$$\begin{aligned}\langle Pv, u\rangle_{\mathcal{H}} &= \langle v, Pu\rangle_{\mathcal{H}} \qquad (u \in \mathcal{D})\\ &= \langle v, Pu\rangle_{L^2(\mathbb{R}^n \setminus B_{R_0}(0))}\\ &= \langle v, \Delta(u|_{\mathbb{R}^n \setminus B_{R_0}(0)})\rangle_{L^2(\mathbb{R}^n \setminus B_{R_0}(0))}\\ &= \langle \Delta v, u|_{\mathbb{R}^n \setminus B_{R_0}(0)}\rangle_{L^2(\mathbb{R}^n \setminus B_{R_0}(0))}\\ &= \langle \Delta v, u\rangle_{\mathcal{H}}\end{aligned}$$

4| <u>Define Hilbert space structure for $\mathcal{D}, \mathcal{D}^\alpha$:</u>

$\|u\|_{\mathcal{D}}^2 = \|u\|_{\mathcal{H}}^2 + \|Pu\|_{\mathcal{H}}^2$ for $u \in \mathcal{D}$

$\|u\|_{\mathcal{D}^\alpha} = \|(P+i)^\alpha u\|_{\mathcal{H}}$ $\alpha \geq 0$

<u>Easy Exercise:</u>

If $\phi \in C_0^\infty(\mathbb{R}^n \setminus B_{R_0}(0))$, $u \in \mathcal{D}^\alpha$, show
$\phi u \in H^{2\alpha}(\mathbb{R}^n)$

<u>Define:</u>

$\mathcal{D}_0 = \mathcal{D} \cap \mathcal{H}_0$

$\mathcal{D}_{loc} = \{u \in \mathcal{H}_{loc} : \chi u \in \mathcal{D} \ \forall \chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi = 1$ on $B_{R_0}(0)\}$

$\mathcal{D}$ dense in $\mathcal{H} \Longrightarrow \mathcal{D}_0$ dense in $\mathcal{H}_0$

<u>Examples:</u>

1. <u>Potential Scattering</u>

$V \in L_0^\infty(\mathbb{R}^n; \mathbb{R})$, $\operatorname{supp} V \in B_{R_0}(0)$.

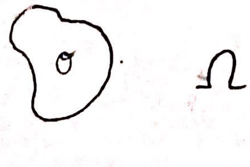
Let $\mathcal{H} = L^2(\mathbb{R}^n)$, $\mathcal{D} = H^2(\mathbb{R}^n)$, $P = \Delta + V$.

(a)–(d) satisfied (Resolvent estimates for (d))

2. <u>Obstacle Scattering (Dirichlet)</u>

$\mathcal{O} \subseteq \overline{B_{R_0}(0)}$ open set, $\Omega = \mathbb{R}^n \setminus \mathcal{O}$

Suppose $\partial\mathcal{O}$ is smooth hypersurface



Let $\mathcal{H} = L^2(\Omega)$

$\mathcal{D} = H^2(\Omega) \cap H_0^1(\Omega)$

$= \{u \in H^2(\Omega) : u|_{\partial\mathcal{O}} = 0\}$

(Can also take <u>Neumann</u> case
$\mathcal{D} = \{u \in H^2(\Omega) : \partial_\nu u|_{\partial\mathcal{O}} = 0\}$)

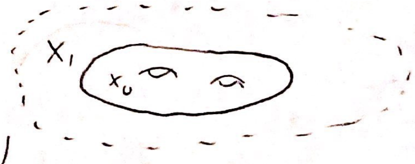
$P = \Delta$

(a)–(d) satisfied (Elliptic regularity for (d))

3. Finite Volume Surface Scattering

(X,g) complete Riemannian surface

$X = X_0 \cup X_1$, $\partial X_0 = \partial X_1$ smooth



$(X_1, g|_{X_1}) = (S^1_\theta \times [a,\infty)_r,\ dr^2 + e^{-2r} d\theta^2)$

Let $\mathcal{H} = \underbrace{L^2(X_0) \oplus \mathcal{H}_a^0}_{\mathcal{H}_a} \oplus L^2([a,\infty), dr)$ with

$\mathcal{H}_a^0 = \{(a_n(r))_{n\in\mathbb{Z}^*} : a_n \in L^2([a,\infty)),\ \sum_{n\in\mathbb{Z}} \int_a^\infty |a_n(r)|^2 dr < \infty$ $\qquad (\mathbb{Z}^* = \mathbb{Z}\setminus\{0\})$

If $u = (u|_{X_0}, (a_n(r))_{n\in\mathbb{Z}^*}, a_0(r))$

projections are:

$\mathbb{1}_{[0,a]} u := (u|_{X_0}, (a_n(r))_{n\in\mathbb{Z}^*})$ $\qquad \mathbb{1}_{[a,\infty)} u := a_0(r)$

Norm is:

$\|u\|_{\mathcal{H}}^2 = \int_{X_0} |u|_{X_0}|^2 d\mathrm{vol}_g + \sum_{n\in\mathbb{Z}} \int_a^\infty |a_n(r)|^2 dr$

Identify $u \in H^1(X)$ with

$(u|_{X_0}, \{e^{-r/2} u_n(r)\}_{n\in\mathbb{Z}^*}, e^{-r/2} u_0(r))$ where $u_n(r) = \frac{1}{2\pi}\int_{S^1} u(r,\theta) e^{-in\theta} d\theta$, $r > a$.

Extend (Fubini's Theorem) to $u \in L^2(X)$, get

$\iota : L^2(X) \to \mathcal{H}$

is an isomorphism

Operator:

$\Delta_g u|_{X_1} = -(e^{-r}\partial_r e^r \partial_r + e^{-2r}\partial_\theta^2) u(r,\theta)$ $\qquad (u \in C_0^\infty(X))$

$\qquad = -\sum_{n\in\mathbb{Z}} (\partial_r^2 - \frac{1}{4} - e^{2r} n^2)(e^{-r/2} u_n(r)) e^{in\theta + r/2}$

Define P as Friedrichs extension via form

$Q_g(u,u) = \sum_{n\in\mathbb{Z}^*} \int_a^\infty (|\partial_r a_n(r)|^2 + n^2 e^{2r} |a_n(r)|^2) dr + \int_a^\infty |\partial_r a_0(r)|^2 dr + \int_{X_0} (|du|_g^2 - \frac{1}{4}|u|^2) d\mathrm{vol}_g$

for $a_n(r) = e^{-r/2} \frac{1}{2\pi} \int_{S^1} u(r,\theta)\, d\theta$

Can check (a)-(d) satisfied (Exercise)

Shows $\mathcal{H}_{R_0}$ can be abstract!

6

Meromorphic Continuation

We want meromorphic continuation for BBH P:

$(P-\lambda^2)^{-1}: \mathcal{H} \to \mathcal{D}$ for $\operatorname{Im}\lambda > 0$, $\lambda^2 \notin \operatorname{spec} P$

to $R(\lambda): \mathcal{H}_0 \to \mathcal{D}_{loc}$ for $\lambda \in \mathbb{C}$.

Two lemmas

* Lemma 1 (Localization Compactness Lemma):

Let P be BBH, $R \geq R_0$, and $\mathbb{1}_{B_R(0)}$ projection onto $\mathcal{H}_{R_0} \oplus L^2(B_R(0) \setminus B_{R_0}(0))$.

Then for $\lambda^2 \notin \operatorname{Spec} P$, $\operatorname{Im}\lambda > 0$,

$\mathbb{1}_{B_R(0)}(P-\lambda^2)^{-1}$, $(P-\lambda^2)^{-1}\mathbb{1}_{B_R(0)}$ are compact operators

Proof:

By closure under adjoints, only do first operator

Case 1: $R = R_0$:

$$\mathbb{1}_{B_{R_0}(0)}(P-\lambda^2)^{-1} = \underbrace{\mathbb{1}_{B_{R_0}(0)}(P-i)^{-1}}_{\text{compact}} + \underbrace{\mathbb{1}_{B_{R_0}(0)}(P-i)^{-1}}_{\text{compact}}\underbrace{(i+\lambda^2)(P-\lambda^2)^{-1}}_{\text{bounded}}$$

Case 2: $R > R_0$:

We have $(P-\lambda^2)^{-1}: \mathcal{H} \to \mathcal{D}$, so

$$(\mathbb{1}_{B_R(0)} - \mathbb{1}_{B_{R_0}(0)})(P-\lambda^2)^{-1}: \mathcal{H} \to H^2(B_R(0)\setminus B_{R_0}(0)) \quad \text{(Condition (a))}$$
$$\hookrightarrow L^2(B_R(0)\setminus B_{R_0}(0)) \quad \text{(Rellich-Kondrachov)}$$

so compact.

* Lemma 2: (Black Box "Free Resolvent Estimate")

If P is BBH, $k \in [0,2]$, $\tau > 0$, then

$$\|\mathbb{1}_{\mathbb{R}^n \setminus B_{R_0}(0)}(P-i\tau)^{-1}\|_{\mathcal{H} \to H^k(\mathbb{R}^n \setminus B_{R_0}(0))} \lesssim \frac{\langle \tau \rangle^{k/2}}{\tau}$$

Proof:

$k=0$: From $\|(P-i\tau)^{-1}\|_{\mathcal{H}\to\mathcal{H}} = \dfrac{1}{d(\lambda, \operatorname{spec} P)}$

$k=2$: Use $P(P-i\tau)^{-1} = I + i\tau(P-i\tau)^{-1}$

2) Then $\|(P-i\tau)^{-1}\|_{\mathcal{H}\to\mathcal{D}} \lesssim \frac{\langle T\rangle}{T}$

<u>$k\in(0,2)$</u>: Interpolate

*<u>Thm</u> (<u>General Meromorphic Continuation</u>)

If P is BBH, n odd, then

$$R(\lambda) := (P-\lambda^2)^{-1} : \mathcal{H} \to \mathcal{D}$$

is meromorphic for $\operatorname{Im}\lambda > 0$, and extends meromorphically to

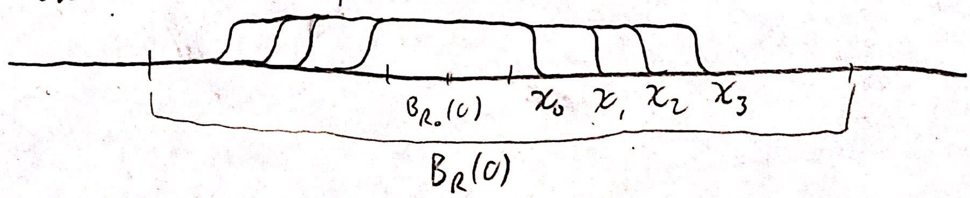
$$R(\lambda): \mathcal{H}_0 \to \mathcal{D}_{loc} \qquad \lambda\in\mathbb{C}$$

<u>Note</u>: For n even, $R(\lambda)$ meromorphic on $\lambda \in \exp^{-1}(\mathbb{C}\setminus\{0\})$

<u>Proof</u>:

<u>1) $\operatorname{Im}\lambda > 0$</u>:

Choose "nested bump functions"



Let λ_0 have $\operatorname{Im}\lambda_0 > 0$, $\lambda_0^2 \notin \operatorname{Spec} P$ to be chosen later

<u>Define</u>: $Q(\lambda,\lambda_0) = Q_0(\lambda) + Q_1(\lambda_0)$ ("approximate inverse" for $P-\lambda^2$)

$$Q_0(\lambda) = (1-\chi_0)R_0(\lambda)(1-\chi_1)$$
$$Q_1(\lambda_0) = \chi_2(P-\lambda_0^2)^{-1}\chi_1$$

Then $(P-\lambda^2)Q_0(\lambda) = 1-\chi_1 + \underbrace{[\Delta,\chi_0]R_0(\lambda)(1-\chi_1)}_{K_0(\lambda)}$

$$(P-\lambda^2)Q_1(\lambda_0) = \chi_1 + \underbrace{(\lambda_0^2-\lambda^2)\chi_2(P-\lambda_0^2)^{-1}\chi_1 + [P,\chi_2](P-\lambda_0^2)^{-1}\chi_1}_{K_1(\lambda,\lambda_0)}$$

<u>Claim</u>: K_0 and K_1 are compact on $\mathcal{H}$

<u>K_0 Proof</u>: $[\Delta,\chi_0]R_0(\lambda)(1-\chi_1): L^2 \to H^2(B_R(0)\setminus B_{R_0}(0))$

K_1 Proof: Use that $\operatorname{supp}\chi_2 \subseteq B_R(0)$, so

$$\mathbb{1}_{B_R(0)}\chi_2 = \chi_2 = \chi_2 \mathbb{1}_{B_R(0)}$$

Apply localization compactness lemma.

Then $K(\lambda,\lambda_0) = K_0(\lambda) + K_1(\lambda,\lambda_0)$ is compact

$$(P-\lambda^2)G(\lambda,\lambda_0) = \underbrace{I + K(\lambda,\lambda_0)}_{\text{Fredholm}}$$

Invert $I+K(\lambda,\lambda_0)$ with Analytic Fredholm theorem, let

$$R(\lambda) = G(\lambda,\lambda_0)(I+K(\lambda,\lambda_0))^{-1}$$

But wait... we need $I+K(\lambda,\lambda_0)$ invertible for <u>some</u> λ.

Choose $\lambda_0 = e^{i\frac{\pi}{4}}\mu$ for μ large

$$\|[P,\chi_2](P-\lambda_0^2)^{-1}\chi_1\|_{\mathcal{H}\to\mathcal{H}} \lesssim \|\mathbb{1}_{\mathbb{R}^n\setminus B_{R_0}(0)}(P-i\mu^2)^{-1}\|_{\mathcal{H}\to H^1(\mathbb{R}^n\setminus B_{R_0}(0))} \quad \text{(Boundedness of } P)$$

$$\lesssim \frac{1}{\mu} \quad \text{("Free Resolvent Estimate")}$$

$$\|[\Delta,\chi_0]R_0(\lambda_0)(1-\chi_1)\|_{\mathcal{H}\to\mathcal{H}} \lesssim \|\mathbb{1}_{\mathbb{R}^n\setminus B_{R_0}(0)}(\Delta-i\mu^2)^{-1}\|_{\mathcal{H}\to H^1(\mathbb{R}^n\setminus B_{R_0}(0))} \quad \text{(Boundedness of } \Delta)$$

$$\lesssim \frac{1}{\mu}$$

Then $\|K(\lambda_0,\lambda_0)\|_{\mathcal{H}\to\mathcal{H}} \lesssim \frac{1}{\mu} \ll 1$

$I+K(\lambda_0,\lambda_0)$ invertible by Neumann series

Then $I+K(\lambda,\lambda_0)$ invertible for $\operatorname{Im}\lambda > 0$

<u>2. General λ:</u>

Again let $\lambda_0 = e^{i\pi/4}\mu$

$$(1-\chi_3)K(\lambda,\lambda_0) = 0$$

$$I + K(\lambda,\lambda_0) = (I + K(\lambda,\lambda_0)(1-\chi_3))(I+K(\lambda,\lambda_0)\chi_3)$$

$$(I+K(\lambda,\lambda_0)(1-\chi_3))^{-1} = I - K(\lambda,\lambda_0)(1-\chi_3)$$

$$\Rightarrow (P-\lambda^2)^{-1} = G(\lambda,\lambda_0)(I+K(\lambda,\lambda_0)\chi_3)^{-1}(I-K(\lambda,\lambda_0)(1-\chi_3)) \quad (\operatorname{Im}\lambda > 0)$$

But $\left.\begin{array}{l}\lambda \longmapsto K_0(\lambda)\chi_3 \\ \lambda \longmapsto K_1(\lambda,\lambda_0)\end{array}\right\}$ meromorphic families of compact operators

Then $\lambda \longmapsto (I + K(\lambda,\lambda_0)\chi_3)^{-1} : \mathcal{H} \to \mathcal{H}$ meromorphic

$\lambda \longmapsto G(\lambda,\lambda_0) : \mathcal{H} \to \mathcal{D}_{loc}$ meromorphic

$\lambda \longmapsto I - K(\lambda,\lambda_0)(1-\chi_3) : \mathcal{H}_0 \to \mathcal{H}_0$ meromorphic

K_0 not bounded on $\mathcal{H}$, but maps $\mathcal{H}_0 \to \mathcal{H}_1$

Then $(P-\lambda^2) = G(\lambda,\lambda_0)(I + K(\lambda,\lambda_0)\chi_3)^{-1}(I - K(\lambda,\lambda_0)(1-\chi_3))$
is meromorphic family, as desired.

<u>Examples:</u>

1. <u>Potential Scattering</u>

Same as before!

2. <u>Obstacle Scattering</u>

$(\Delta_0 - \lambda^2)^{-1} : L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$

continues meromorphically.

For $\operatorname{Im}\lambda > 0$, can write

$(\Delta_0 - \lambda^2)^{-1} u(x) = \int_\Omega G(\lambda,x,y) u(y)\,dy$ for <u>Green's function</u> G.

We get $\lambda \mapsto G(\lambda,x,y)$ (for $x \neq y$) is meromorphic.

3. <u>Finite Volume Surface Scattering</u>

$(\Delta_g - \frac{1}{4} - \lambda^2)^{-1} : L^2(X) \to H^2(X)$ $\operatorname{Im}\lambda > 0$

extends meromorphically to

$R(\lambda) : L^2_0(X) \to H^2_{loc}(X)$

Can see that

$\mathrm{Spec}_{cont}(\Delta_g) = [\frac{1}{4}, \infty)$ $\mathrm{Spec}_{pp}(\Delta_g) = E_1, \ldots, E_N, z_1, z_2, \ldots$

$0 = E_0 < E_1 \leq \ldots \leq E_N \leq \frac{1}{4} < z_1 \leq z_2 \leq \ldots$

<u>Other Analogous Results</u>

Def: Let P be BBH, n odd, $\lambda \in \mathbb{C}$.
Poles of $R(\lambda)$ are called <u>resonances</u> and the <u>multiplicity</u> of λ is

$$m_R(\lambda) := \operatorname{rank} \oint_{\gamma_\lambda} R(\zeta)\, d\zeta$$

Note $m_R(\lambda) = m_R(-\bar{\lambda})$ as before

***Thm** (<u>Singular Part of Resolvent</u>)
If $m_R(\lambda) > 0$, $\lambda \neq 0$, there is $M_\lambda \leq m_R(\lambda)$ s.t.

$$R(\zeta) = -\sum_{k=1}^{M_\lambda} \frac{(P-\lambda^2)^{k-1}}{(\zeta^2-\lambda^2)^k} \Pi_\lambda + \underbrace{A(\zeta,\lambda)}_{\substack{\text{holomorphic}\\ \text{near } \lambda}}$$

$$\Pi_\lambda = -\frac{1}{2\pi i}\oint_\lambda R(\lambda)\, 2\zeta\, d\zeta, \quad m_R(\lambda) = \operatorname{rank} \Pi_\lambda, \quad (P-\lambda^2)^{M_\lambda}\Pi_\lambda = 0$$

<u>If $u \in \Pi_\lambda(\mathcal{H}_0)$, $(P-\lambda^2)u = 0$, then u is called resonant state</u>
If $u \in \Pi_\lambda(\mathcal{H}_0)$, u is <u>generalized resonant state</u>
(<u>Note</u>: u <u>not necessarily</u> an eigenfunction, as need only be in $\mathcal{D}_{loc}$)

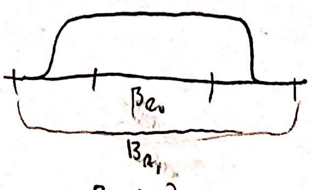
***Thm** (<u>Characterization of Resonant States</u>)
A function $u \in \mathcal{D}_{loc}$ is resonant state for $\lambda \in \mathbb{C} \setminus \{0\}$ iff

$$(P-\lambda^2)u = 0, \quad \underbrace{u|_{\mathbb{R}^n \setminus B_R(0)} = R_0(\lambda) g|_{\mathbb{R}^n \setminus B_R(0)}}_{\text{"Outgoing" condition}} \quad \text{for some } g \in L^2_0(\mathbb{R}^n),\ R > 0.$$

<u>Resonance Upper Bounds</u>

Let $\mathbb{T}^n_{R_1} = \mathbb{R}^n / R_1\mathbb{Z}$ for $R_1 > R_0$

$$\mathcal{H}^\#_{R_1} := \mathcal{H}_{R_0} \oplus L^2(\mathbb{T}^n_{R_1} \setminus B_{R_0}(0))$$



If $P: \mathcal{H} \to \mathcal{H}$ is BBH with domain $\mathcal{D} \subseteq \mathcal{H}$,

$$\mathcal{D}^\#_{R_1} = \{u \in \mathcal{H}^\#_{R_1} : \chi u \in \mathcal{D},\ (1-\chi)u \in H^2(\mathbb{T}^n_{R_1})\ \forall \chi \in C_0^\infty(B_{R_1}(0)), \text{ with } \chi = 1 \text{ near } B_{R_0}(0)\}$$

For any such χ:

The <u>reference operator</u> $P^\#_{R_1} : \mathcal{D}^\#_{R_1} \to \mathcal{H}^\#_{R_1}$ is

$$\boxed{P^\#_{R_1} u = P(\chi u) + \Delta((1-\chi)u)}$$

(By (a), (b) independent of χ)

<u>Properties:</u>

1. $P^\# : \mathcal{H}^\# \to \mathcal{H}^\#$ self-adjoint with domain $\mathcal{D}^\#$
2. $(P^\# + i)^{-1}$ compact (so $\operatorname{spec}(P^\#)$ discrete)

<u>Assume the following</u>

$$|\operatorname{Spec}(P^\#) \cap \{|z| \leq r\}| \lesssim r^{n^\#} \quad \text{for some } n^\# \geq n \qquad (*)$$

<u>Examples:</u>

1. <u>Elliptic Perturbation</u>

$$Pu = \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x) D_{x_j} u) + c(x)u$$

with $a_{ij} - \delta_{ij} \in C_0^\infty(B_{R_0}(0); \mathbb{R})$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j \geq c_0 |\xi|^2 \quad \forall \xi \in \mathbb{R}^n$$

Then P is BBH, and by Generalized Weyl law

$n^\# = n$.

2. <u>Finite-Volume Surfaces</u>

$\mathcal{H} = \mathcal{H}_a \oplus L^2([a, \infty))$

$\mathcal{H}^\#_b := \mathcal{H}_a \oplus L^2([a, b])$

Here $n=1$, but $n^\# = 2$.

*Thm (<u>Black Box Resonance Upper Bound</u>)

If P is BBH with $n \geq 1$ odd, and $(*)$ holds, then

$$\sum_{|\lambda| \leq r} m_R(\lambda) \lesssim r^{n^\#}.$$